The Limit Superior and Limit Inferior

A number a is called a **limit point** of the sequence $\{a_n\}$ if it is the limit of a subsequence of $\{a_n\}$. A bounded sequence has at least one limit point according to Bolzano-Weierstrass Theorem. A properly divergent sequence does not have any limit point.

Let $\{a_n\}$ be a sequence bounded from below. For each $k \geq 1$, the number

$$\beta_k = \sup_{n \geq k} a_n = \{a_k, a_{k+1}, a_{k+2}, \cdots\} \ ,$$

is in $(-\infty, \infty]$. It is clear that $\{\beta_k\}$ is decreasing and bounded from below. By Monotone Convergence Theorem, its limit exists. We call it the **limit superior** of the sequence of $\{a_n\}$. In notation,

$$\overline{\lim}\ a_n, \text{ or } \limsup\{a_n\} = \lim_{k\to\infty} \beta_k = \inf\{\beta_k\} = \inf_k \sup\{a_n\}_{n \geq k} \ .$$

Similarly, the number

$$\alpha_k = \inf_{n \geq k} a_n = \inf\{a_k, a_{k+1}, a_{k+2}, \cdots\} \ ,$$

is a real number when the sequence is bounded from above. It is clear that $\{\alpha_k\}$ is increasing and bounded from above. By Monotone Convergence Theorem, its limit exists. We call it the **limit inferior** of the sequence of $\{a_n\}$. In notation,

$$\underline{\lim}\ a_n, \text{ or } \liminf\{a_n\} = \lim_{k\to\infty} \alpha_k = \sup\{\alpha_k\} = \sup_k \inf\{a_n\}_{n \geq k} \ .$$

Theorem 1. *Let* $a = \overline{\lim}_{n\to\infty} a_n$.

(a) *For each* $\varepsilon > 0$*, there is some* n_0 *such that* $a_n \leq a + \varepsilon$ *for all* $n \geq n_0$.

(b) *For each* $\varepsilon > 0$*, there is a subsequence* $\{a_{n_j}\}$ *satisfying* $a_{n_j} \geq a - \varepsilon$.

Proof. (a) By the definition of infimum, for any $\varepsilon > 0$, there is some k_0 such that $\beta_k \leq \alpha + \varepsilon$ for all $k \geq k_0$. It follows from the definition of β_k that $a_n \leq a + \varepsilon$ for all $n \geq k_0$. It suffices to take $n_0 = k_0$.

(b) It suffices to show there is a subsequence converging to a. Since $a = \lim_{k\to\infty} \beta_k = \inf_k \beta_k$, for each $N \geq 1$, there is some $n(N)$ such that

$$a + \frac{1}{N} > \beta_{n(N)} \geq a \ . \tag{1}$$

From the definition of the supremum, we can find a_{n_N} from $\{a_{n(N)}, a_{n(N)+1}, a_{n(N)+2}, \cdots\}$ to form a subsequence $\{a_{n_N}\}$ such that

$$\beta_{n(N)} \geq a_{n_N} > \beta_{n(N)} - \frac{1}{N} \ . \tag{2}$$

Combining (1) and (2), we have

$$|a_{n_N} - a| < \frac{1}{N} \ .$$

It follows that the subsequence $\{a_{n_N}\}_{N=1}^{\infty}$ converges to a.

From Theorem 1, we deduce the following characterization of limit superior and limit inferior.

Theorem 2. *The limit superior of a bounded sequence is its largest limit point and its limit infimum is its smallest limit point.*

As an application to power series, we prove

Theorem 3 (Cauchy-Hadamard) *The power series $\sum a_n x^n$ is absolutely and uniformly convergent on $[-r, r]$ for $r \in (0, R)$ where R is its radius of convergence, and it is divergent at any $x, |x| > R$.*

We have taken the center $x_0 = 0$ for simplicity. Recall that a series of functions $\sum f_n$ is called absolutely and uniformly convergent on some set E if $\sum_{k=1}^{\infty} |f_k|(x)$ is uniformly convergent on E. It implies that $\sum_{k=1}^{\infty} f_k(x)$ is also uniformly convergent on E.

Proof. Recall that $R = 1/\rho$ where $\rho = \overline{\lim}_{n\to\infty} |a_n|^{1/n} \in [0, \infty]$. According to Theorem 1(a), for each $\varepsilon > 0$, $|a_n|^{1/n} \le \rho + \varepsilon$ for all $n \ge n_0$. As a result,

$$(|a_n||x|^n)^{1/n} = |a_n|^{1/n}|x| \le r|a_n|^{1/n} \le r(\rho + \varepsilon), \quad \forall x \in [-r, r] \ , n \ge n_0 \ .$$

Observing that $r(\rho+\varepsilon) < 1$ when $\varepsilon = 0$, we can find a small $\varepsilon_0 > 0$ such that $r_0 \equiv r(\rho+\varepsilon_0) < 1$. It follows that

$$|a_n||x|^n \le r_0^n \ , \quad \forall n \ge n_0 \ .$$

By M-Test, $\sum a_n x^n$ converges absolutely and uniformly on $[-r, r]$.

On the other hand, for each $\varepsilon > 0$, there is a subsequence a_{n_n} satisfying $a_{n_j} \ge a - \varepsilon$. Therefore, $|a_n x^n|^{1/n} = |x||a_n|^{1/n} \ge |x|(\rho - \varepsilon)$ at all $n = n_j$. Since $|x|\rho > 1$, we can fix a small ε_1 such that $|x|(\rho - \varepsilon_1) \ge 1$, so $|a_n x^n| \ge 1$ at all $n = n_j$. It implies that $\sum a_n x^n$ is divergent (since $a_n x^n$ must converge to 0 when it is convergent).